ABSTRACTS FOR THE SCHOOL ON HIGGS BUNDLES AND FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES

Lecture Courses

Bruno Klingler: Fundamental groups of smooth projective complex varieties. The homotopy type of a smooth projective irreducible complex variety (or more generally: of a compact Kähler manifold) is strongly constrained. Its most mysterious part lies in the fundamental group. The main question in this area, attributed to Serre, consists in *characterising the fundamental groups of smooth projective irreducible complex varieties (the so-called projective groups) among finitely presented groups.* Many constraints come from Hodge theory. If X is a smooth projective irreducible complex variety and Γ its fundamental group then classical Hodge theory on X forces strong restrictions on Γ. For example the Hodge decomposition $H^1(X', \mathbb{C}) = H^0(X', \Omega^1_{X'}) \oplus \overline{H^0(X', \Omega^1_{X'})}$ for any finite étale cover X' of X implies that any finite index subgroup of Γ has necessarily an even first Betti number. Thus a free group is never a projective group. During the 90's Hitchin, Corlette and Simpson developed non-abelian Hodge theory, which can be thought as a vast generalization of the usual Hodge theory for $H^1(X, \mathbb{C})$: namely an interpretation of the first cohomology $H^1(X, \mathcal{GL}(n, \mathbb{C}))$ (the moduli space of representations of Γ into $\mathbf{GL}(n, \mathbb{C})$) in terms of holomorphic objects on X (semi-stable Higgs bundles with vanishing Chern classes). A remarkable corollary of this theory is the proof by Simpson that the group $SL(n, \mathbb{Z})$, $n \geq 2$, is not a projective group. After a general introduction to projective groups, these lectures will be devoted to non-Abelian Hodge theory and its consequences, in particular the relation between rigidity properties for representations of Γ and the existence of symmetric algebraic differentials on X. As an application, we will prove using automorphic forms some rigidity properties for the fundamental groups of some Shimura varieties of $U(n, 1)$-type.

Adrian Langer: Vector bundles and Higgs bundles in positive characteristic. The main aim of the lectures is to show some recent applications of positive characteristic version of non-abelian Hodge theory to study of algebraic varieties and their invariants in positive and zero characteristics. Approximate content of the lectures (subject to change) is the following:

Lecture I. Semistability in positive characteristic, Cartier descent, instability of Frobenius pull backs, bounding number of sections

Lecture II. Spreading out, Bogomolov's inequality in characteristic zero and in positive characteristic, various counterexamples, Witt ring and lifting

Lecture III. Ogus–Vologodsky's version of Simpson's correspondence in positive characteristic ("easy" version after Lan-Sheng-Zuo). Logarithmic version of the correspondence. Proof of Langton's theorem and proof of Lan-Sheng-Zuo's conjecture on existence of Higgs–de Rham flows for semistable Higgs bundles.

Lecture IV. Bogomolov's inequality for semistable Higgs bundles in characteristic zero, Bogomolov's inequality for semistable Higgs bundles in positive characteristic, Bogomolov–Miyaoka–Yau inequality in positive characteristic, semistability of tensor products and restriction theorems.

Research Talks

Oscar Garcia-Prada: Action of the mapping class group on character varieties and Higgs bundles. Abstract: We consider the action of the mapping class group of a compact surface S of genus $g > 1$ on the character variety of the fundamental group of S in a connected semisimple real Lie group G. We identify the fixed points of the action of a finite subgroup Γ of the modular group on the character variety, in terms of G-Higgs bundles equipped with a Γ-equivariant structure on a Riemann surface $X = (S, J)$, where J is an element in the Teichmüller space of S for which Γ is included in the group of automorphisms of X, whose existence is guaranteed by Kerckhoff's solution of the Nielsen realization problem. The Γ-equivariant G-Higgs bundles are in turn in correspondence with parabolic Higgs bundles on $Y = X/\Gamma$, where the weights on the parabolic points are determined by the Γ-equivariant structure. This generalizes work of Nasatyr & Steer for $G = SL(2, \mathbf{R})$, and Andersen & Grove and Furuta & Steer for $G = SU(n)$.

Yohan Brunebarbe: Level structures on abelian varieties over complex functions fields. I will discuss the following result: Let X be a smooth projective complex variety. Assume that a non-empty Zariski-open subset U of X parametrizes a family of principally polarized abelian varieties of dimension g with a level-n structure whose variation is maximal. Then X is of general type as soon as $n > 12g$. A key input in the proof is the existence on U of a Higgs bundle of a special kind.

Julien Keller: Higgs bundles and quantization of Hitchin equation. Abstract: We provide an algebraic framework for geometric quantization of Hermitian metrics that are solutions of Hitchin equation for Higgs bundles over a projective manifold. More precisely, using Geometric Invariant Theory, we introduce a notion of balanced metrics in this context. Balanced metrics are canonical algebraic objects. We show that balanced metrics converge at the quantum limit towards the solution of the Hitchin equation. We relate the existence of balanced metrics to the Gieseker stability of the Higgs bundle.